

# Anti-Viral 1 Solutions

## 1 Problems

### Problem 1-1

What is the **largest positive integer** that can be made with five 3s and **one each** of the four basic arithmetic operations (i.e.  $+$ ,  $-$ ,  $\times$ ,  $\div$ )? (No brackets, exponents, concatenation, etc allowed)

### Solution 1-1

By testing a variety of combinations (and noting that we cannot use brackets), we see that

$$3 \times 3 + 3 - 3 \div 3 = \boxed{11}$$

is the maximum possible value we can achieve.

### Problem 1-2

We normally write our numbers in **base 10**; that is, each of our 10 digits represents powers of 10, so that, for example,  $157 = 7 \cdot 10^0 + 5 \cdot 10^1 + 1 \cdot 10^2$ . Numbers can be written in other bases as well: computers use base 2, or binary, so each digit represents a power of 2:

$$\begin{aligned} 10011101_2 &= 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 + 1 \cdot 2^4 + 0 \cdot 2^5 + 0 \cdot 2^6 + 1 \cdot 2^7 \\ &= 157 \end{aligned}$$

Note that a subscript 2 is used to remind us that this number is in **base 2**.

What is the one-hundredth smallest positive integer that can be written using no digits other than 0, 1, and 2 in base 4?

### Solution 1-2

We first ignore the base 4 condition and find the 100<sup>th</sup> smallest positive integer that can be written using only 0, 1, and 2.

Listing them out (1, 2, 10, 11, 12, 20, 21, 22, 100, 101, 102, 110, etc), we notice that we are essentially just counting in base 3.

The 100<sup>th</sup> smallest positive integer in base 3 is just 100<sub>10</sub>, so we get

$$100 = 81 + 2 \cdot 9 + 1 \implies 100 = 10201_3$$

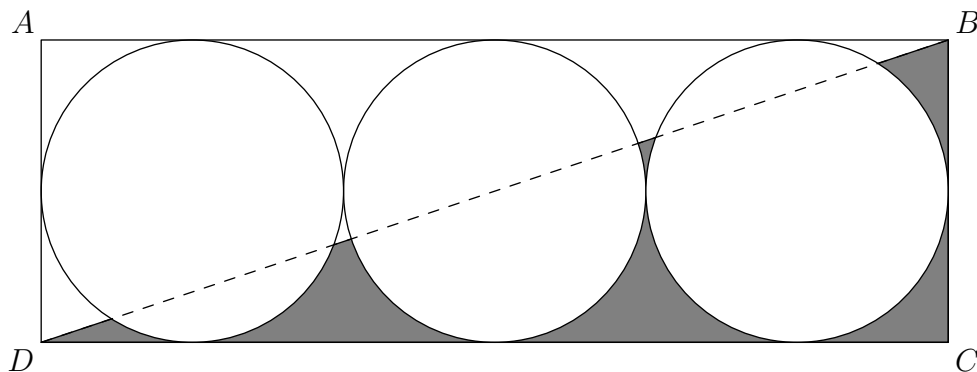
Now we convert our answer from base 3 to base 4 to get

$$10201_3 = 256 + 2 \cdot 16 + 1 = \boxed{289}$$

*Note:* Due to the unclear problem statement,  $\boxed{10201}$  was also accepted.

**Problem 1-3**

Three identical circles of radius 2 are externally tangent and inscribed within a rectangle of length 12 and width 4. Note that the shaded region represents the the portion of triangle  $BCD$  that does not overlap with the circles. The area of this shaded region can be represented as  $m - n\pi$ , where  $m$  and  $n$  are positive integers. Find  $m + n$ .

**Solution 1-3**

We notice that the diagram is (basically) symmetrical and that the area within  $\triangle BCD$  that is not contained within any of the circles is the same as the area within  $\triangle DAB$  that is not contained within any of the circles.

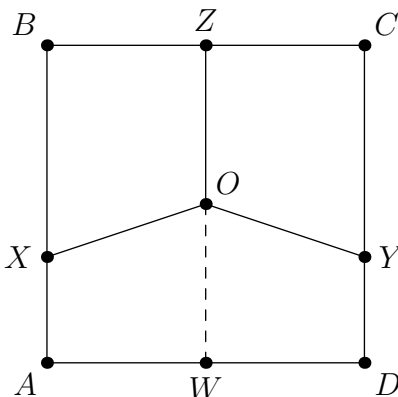
Therefore, the shaded area is just half the area of the rectangle minus the area of the circles:

$$\begin{aligned}
 A_s &= \frac{A_{\square} - 3A_{\circ}}{2} \\
 &= \frac{48 - 12\pi}{2} \\
 &= 24 - 6\pi \\
 \implies m &= 24, n = 6 \\
 m + n &= \boxed{30}
 \end{aligned}$$

Another way to see this is to notice that the two small shaded regions above the midline of  $BC$ , when rotated, fit exactly into the unshaded region below the midline.

**Problem 1-4**

In the diagram below (diagram not to scale),  $ABCD$  is a square,  $Z$  is the midpoint of  $BC$ ,  $OZ \perp BC$  and  $OZ = \frac{1}{2}AB$ .  $X$  and  $Y$  are on  $AB$  and  $CD$ , respectively, such that  $AX = DY$ . It is known that the three enclosed regions  $ADYOX$ ,  $BXOZ$ ,  $CZOY$  have the same area. The value of  $\frac{AX}{XB}$  can be expressed in the form  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

**Solution 1-4**

Extend  $ZO$  to meet side  $AB$  at point  $W$ .

Then, the area of trapezoid  $BXOZ$  should be twice that of trapezoid  $AXOW$ .

We have that

$$A_{BXOZ} = \frac{BZ(BX + ZO)}{2}$$

$$A_{AXOW} = \frac{AW(AX + WO)}{2}$$

We also know that

$$BZ = AW = ZO = WO = \frac{BX + AX}{2}$$

Setting  $A_{BXOZ} = 2A_{AXOW}$ , we get that

$$2 \left( \frac{3}{2}AX + \frac{1}{2}BX \right) = \frac{1}{2}AX + \frac{3}{2}AX$$

$$\frac{5}{2}AX = \frac{1}{2}BX$$

$$\frac{AX}{BX} = \frac{1}{5}$$

$$\implies m = 1, n = 5$$

$$m + n = \boxed{6}$$

Alternatively, let  $AB = 1$ . Then, the area of  $BXOZ$  is  $1/3$  and the area of  $XOAW$  is  $1/6$ .

Solving with  $BZ = ZO = OW = WA = 1/2$ , we get that

$$\begin{aligned} AX &= \frac{1}{6}, BX = \frac{5}{6} \\ \frac{AX}{BX} &= \frac{1}{5} \\ \implies m &= 1, n = 5 \\ m + n &= \boxed{6} \end{aligned}$$

### Problem 1-5

How many tetrahedra with positive volume can be made using 4 distinct vertices of a cube?

### Solution 1-5

We first note that each tetrahedron has four points, and thus there are

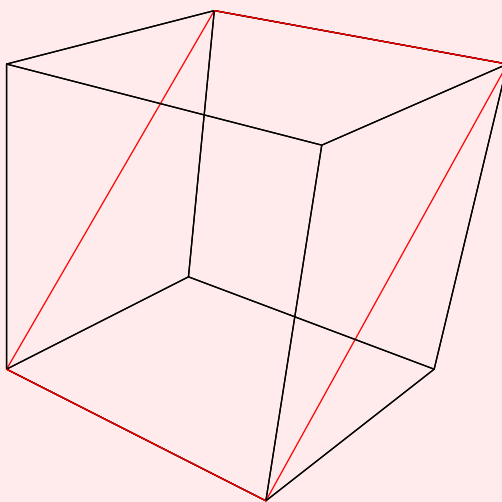
$$\binom{8}{4} = 70$$

ways to form a tetrahedron. However, some of these tetrahedra have zero volume.

In order for a tetrahedron to have zero volume, it must be flat (i.e. a plane). Since we only need to consider a  $2 \times 2 \times 2$  cube, our options are extremely limited.

There are obviously six (flat) faces to the cube.

Then, we also notice that there are six "diagonal faces"



So our final answer is just

$$70 - 6 - 6 = \boxed{58}$$

**Problem 1-6**

Let  $a$  and  $b$  be integers, such that  $\log_{\sqrt{2}} a$  is rational. If

$$\log_{\sqrt{2}} a + \log_a b = \frac{\log_{\sqrt{2}} b}{2},$$

then compute the sum of all possible values of  $a + b$ .

**Solution 1-6**

Let  $\log_{\sqrt{2}} a = r$ , where  $r$  is a rational number.

Since  $a = \sqrt{2}^r$  is an integer,  $r$  must be an even integer.

In other words, we can let  $a = 2^n, n \in \mathbb{N}$  (note that  $n \neq 0$  since 1 is not a valid base)

We can substitute and use the log base change formula to simplify our expression:

$$\begin{aligned} \log_{\sqrt{2}} a + \log_a b &= \frac{\log_{\sqrt{2}} b}{2} \\ \log_{\sqrt{2}} (2^n) + \log_{(2^n)} b &= \frac{\log_{\sqrt{2}} b}{2} \\ 2n + \frac{\log_2 b}{n} &= \log_2 b \\ \log_2 b &= \frac{2n^2}{n-1} \\ b &= 2^{\left(\frac{2n^2}{n-1}\right)} \end{aligned}$$

Therefore, we need  $\frac{2n^2}{n-1}$  to be an integer. However, note that  $\gcd(n^2, n-1) = 1$  for all  $n$ , and 2 is not divisible by  $n-1$  for  $n > 3$ .

Therefore, the only solutions are  $n = 2, 3$ , which gives the pairs

$$(4, 256), (8, 512) \implies 4 + 256 + 8 + 512 = \boxed{780}$$