Anti-Viral 2 Solutions

1 Problems

Problem 2-1

A **chameleon** is a 4-digit positive integer with distinct digits (and no leading zeroes). What is the smallest chameleon that is divisible by 9?

Solution 2-1

Note that the smallest chameleon is 1023, as no leading zeroes are allowed.

Since the remainder when 1023 is divided by 9 is 6, we need to add at least 3 to get a number divisible by 9. Since $1023 + 3 = 1026$ has distinct digits, our answer is just $\boxed{1026}$.

Problem 2-2

Trish has four types of socks in her drawer, in the following quantities: 3 blue polka-dot, 5 red polka-dot, 4 blue striped, and 3 red striped. She's not too fussy about matching: she'll wear any two socks as long as they're both blue **or** both polka-dot. If Trish has to pick her socks in the dark (and thus never knows what socks she has picked out), how many will she have to take out before she's sure she has a pair that she can wear?

Solution 2-2

We can alter our question to: what is the maximum number of socks Trish needs to take before being guaranteed a pair that satisfies the given criteria?

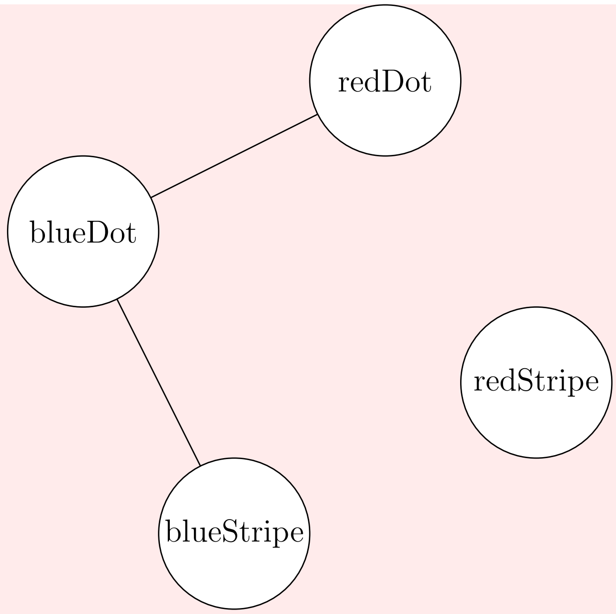
Let us consider how doing each of the options restricts our other options. This influence is bidirectional (i.e. if picking option 1 prevents us from picking option 2, then picking option 2 also prevents us from picking option 1).

Consider the following table:

	Blue	Red
Polka-dot	3	5
Striped	4	3

To maximize the number of pairs, we pick the three red striped socks, then one each of blue striped and red polka-dot. Our next sock guarantees a pair, so our answer is $3+1+1+1 = \boxed{6}$.

An elegant way to approach problems like these is with the aid of graph theory. Let's construct a graph where every node represents a type of sock, and the size of that node is the number of socks of that type. Let's add an edge between two nodes if the sock types form a valid pair (both polka-dot or both blue). This will form a graph that looks like:



Now we can translate the problem to finding the size of the maximum independent set, since any adjacent nodes selected would mean there is a possible pair. In our graph, the maximum independent set is $\{\text{redDot}, \text{ blueStripe}, \text{ redStripe}\}$, and its size is $5 + 4 + 3 = 12$.

However, you cannot choose 2 of the same sock from blue stripes or red polka-dots, since they will form a pair with themselves. Therefore, the size of the maximum independent set is actually $1 + 1 + 3 = 5$. Note that this is the maximum number of socks taken without a wearable pair.

Therefore, our result is this value +1, or

$$5 + 1 = \boxed{6}$$

Fun side note: The maximum independent set problem is actually NP-hard, which basically means it is extremely slow to solve for a relatively large number of nodes. However, there are only 4 nodes in the graph in the problem, so the problem is quite solvable, even without computer assistance or fancy algorithms.

Problem 2-3

a, b, and c are three positive integers such that:

- $a, b, c \geq 15$;
- a and b have no common factors (other than 1);
- $a^2 + b^2 = c^2$ (i.e. a, b and c are a **Pythagorean triple**);
- $c = b + 2$.

What is the **least** possible value of c?

Solution 2-3

We can substitute $c = b + 2$ into the equation $a^2 + b^2 = c^2$ to get that

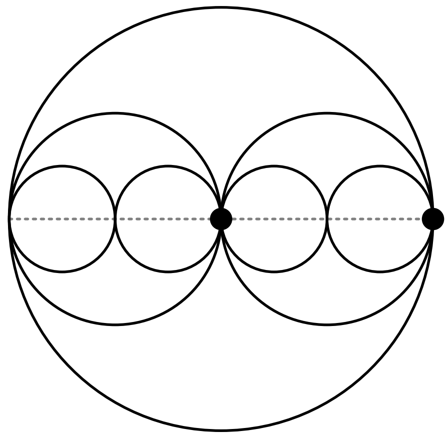
$$a^2 + b^2 = (b+2)^2$$
$$a^2 = 4b + 4$$
$$b = \frac{a^2}{4} - 1$$

Since b is an integer, we can see that a^2 must be a multiple of 4, or a must be even.

Plugging in $a = 16$ (since $a \geq 15$, a even), we get the triple $(16, 63, 65) \implies \boxed{65}$

Problem 2-4

Starting with a circle of radius 4, we inscribe a two circles of radius 2 such that the diameters of the two small circles lie on the same line. We then inscribe two circles in each of the 2 circles in a similar manner. We continue this process indefinitely, and find that the sum of the areas of the circles we draw is $k\pi$ for some integer k. What is the value of k?



Solution 2-4

In the n^{th} step, we draw 2^{n-1} circles, each of radius $\frac{4}{2^{n-1}}$. Thus, the total area is

$$\begin{aligned} A &= \sum_{i=1}^{\infty}\left(2^{n-1}\cdot\pi\cdot\left(\frac{4}{2^{n-1}}\right)^2\right) \\ &= \sum_{i=1}^{\infty}\left(\frac{16\pi}{2^{n-1}}\right) \\ &= 16\pi\cdot 2 \\ &= 32\pi \end{aligned}$$

Thus, $k = \boxed{32}$

We start with 1 circle of radius 4 and area 16π. After every iteration, the number of new circles drawn doubles, but the radius is halved.

Since the radius is squared when calculating area, dividing it by 2 actually reduces it by a factor of $4\left(\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}\right)$.

Then, we multiply by 2 to account for double the amount of circles, so $\frac{1}{4}\cdot 2 = \frac{1}{2}$.

Therefore, the total area drawn after each iteration is halved, and our final answer is

$$16\pi + 8\pi + 4\pi + \cdots = 16\pi\cdot 2 = 32\pi$$

Thus, $k = \boxed{32}$

Problem 2-5

You are given a 5 by 5 grid. You place the numbers 1 to 25 in each cell such that for every cell, if there exists a cell below it, then the number in that cell is greater than that of the original cell. Likewise, if there exists a cell to the right of it, then that cell has a number greater than that of the original cell. Out of all the possible placements of the 25 numbers, how many distinct cells can the number 18 be placed in?

For example, 18 can be placed in the cell at the intersection of the 4th row and 4th column as follows:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	19
15	16	17	18	20
21	22	23	24	25

In contrast, the following arrangement does not work:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
18	19	17	16	20
21	22	23	24	25

Proposed by: Edward Xiao

Solution 2-5

For any cell, all cells up and to the left of it must be less than that number, and all cells down and to the right of it must be greater than that number.

Therefore, we can have at most 17 cells up and to the left of our cell and at most 7 cells down and to the right of our cell.

This gives a total of $\boxed{11}$ possible positions for 18.

Here is a visual representation of the cells 18 can be in (0 = no, 1 = yes):

0	0	0	0	1
0	0	0	1	1
0	0	0	1	1
0	1	1	1	0
1	1	1	0	0

The examples given below are provided for reference. Alternatively, we can use symmetry and only check a couple of cases.

1	2	3	4	18
5	6	7	8	22
9	10	11	12	23
13	14	15	16	24
17	19	20	21	25

1	2	3	16	17
4	5	6	18	22
7	8	9	19	23
10	11	12	20	24
13	14	15	21	25

1	2	3	4	17
5	6	7	8	18
9	10	11	12	23
13	14	15	16	24
19	20	21	22	25

1	2	3	16	21
4	5	6	17	22
7	8	9	18	23
10	11	12	19	24
13	14	15	20	25

1	2	3	4	16
5	6	7	8	17
9	10	11	12	18
13	14	15	19	24
20	21	22	23	25

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	18	19	20	21
17	22	23	24	25

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

1	2	3	4	5
6	7	8	9	10
11	12	13	14	19
15	16	17	18	20
21	22	23	24	25

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	19	20	21
18	22	23	24	25

1	2	3	4	5
6	7	8	9	10
11	12	13	14	19
15	16	20	21	22
17	18	23	24	25

1	2	3	4	21
5	6	7	8	22
9	10	11	12	23
13	14	15	19	24
16	17	18	20	25

Problem 2-6

We consider a modified *Game of Life*. We begin with a certain amount of live cells on an infinite square grid, and every turn the following modifications are implemented:

a) If a cell is **orthogonally** adjacent (a.k.a. shares a side) to **at least two** live cells, then it becomes live in the next turn.

b) All other cells become dead in the next turn.

For how many positive integers n at most 1970, does there exist an initial configuration of n live cells such that this configuration stays constant after each turn (no cells are ever changed from live to dead or vice versa)?

Solution 2-6

Define a *connected component* as a set of cells such that:

a) All cells are live.

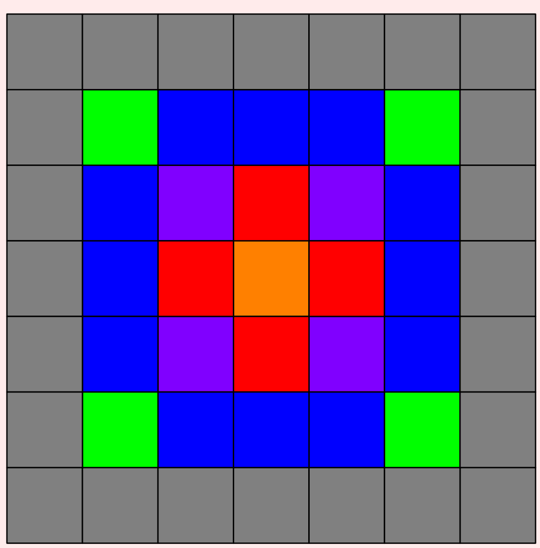
b) All cells are either orthogonally or diagonally connected to some other cell in the set.

c) No other cells are connected to cells in the set.

Note that this is a different definition from what you would usually get with a graph-theoretic interpretation.

We now claim that **all connected components are rectangles of length and width at least 2.** Before we begin, note that all squares that are connected to at least two live cells must be initially live, and any other squares must be initially dead.

To prove this, first consider the "border" of the connected component (the grey cells in the diagram). Note that because of the condition, we have that if two cells are diagonally connected, then the two cells adjacent to each of the cells must also be in the set. Thus, no two cells on the border are diagonally connected, and there are no L shapes. This actually just implies that the border has to be rectangular, because diagonal connections are not allowed and the shape must be convex (or else has an L).



Now, consider the four corners of the rectangle (the green cells in the diagram). These are both connected to two live cells and thus are live themselves. This fills in the "second layer" of the rectangle (the blue squares) as all live.

We can now continue this process until the entire rectangle is filled. Thus, all connected components are rectangles. Finally, note that no 1 by n or n by 1 rectangles can exist (or else the endpoints will die off), so all these rectangles must have a length and width of at least 2 cells.

Now, note that all configurations are made up of disjoint connected components. This means all possible values of the number of cells can be described as:

$$A_1 + A_2 + \dots A_n$$

where A_i can be expressed in the form $m \cdot n$, where m, $n \geq 2$. Note that all even numbers at least 4 are possible values for A_i, and the two smallest odd numbers that are possible are 9 and 15. This indicates that the only possible values for the number of cells are:

a) Any even number at least 4

b) 9

c) Any odd number at least 13 (obtained by adding 4, 6, ... to 9). The fact that 15 is the second-smallest odd number means that we cannot obtain 11.

This means that only 6 positive integers cannot be the number of cells: 1, 2, 3, 5, 7, and 11. Thus, our answer is $1970 - 6 = \boxed{1964}$.

Note: The above solution may seem so long and intimidating that it should be deemed too hard for an Antiviral problem. While this problem is quite difficult, the Exec believed that there were a few aspects of the contest that made solving this problem more realistic, namely:

a) The above solution is long mostly because it's kind of difficult to word a lot of it. Really, the main (albeit tricky) idea is to realize that the entire diagram is just a bunch of rectangles.

b) In addition, the short answer nature on the contest meant that the contestants also did not need to prove their findings, making this mostly just a problem about observation.