Anti-Viral 3 Solutions

1 Problems

Problem 3-1

Glenn bought a collection of Superman figurines. Each individual figurine costs the same amount of money, and Glenn paid \$224 in total. Unfortunately, one of the figurines has been chewed up by his dog, so he sold all of the remaining figurines for \$4 more each than he originally paid for them. Glenn managed to break even — he got back \$224. How many figurines did he originally buy?

Proposed by: Nicole Fan

Solution 3-1

Let f be the number of figurines Glenn has and let p be the price of each figurine. Since the total cost of the figurines was \$224, we have that

$$f \cdot p = 224$$

Since having one less figurine and marking up each of the other figurines by \$4 still gives a total cost of \$224, we see that

$$\begin{aligned} 224 = (f-1)\cdot(p+4) &= f\cdot p + 4f - p - 4 \\ 224 &= 224 + 4f - p - 4 \\ p &= 4f - 4 \end{aligned}$$

Plugging this into our first equation, we have that

$$\begin{aligned} f\cdot(4f-4) &= 224 \\ f^2 - f &= 56 \\ (f-8)(f+7) &= 0 \\ f &= 8 \text{ or } -7 \end{aligned}$$

Since $f > 0$, Glenn originally bought $\boxed{8}$ figurines.

Problem 3-2

Let x be a real number. If $x^3 + 9x^2 + 3x + 3 = 1933$, what is the value of $5x^4 + 3x^2 + 9x + 8$?

Proposed by: Kat Dou

Solution 3-2

Let $f(x) = x^3 + 9x^2 + 3x + 3$, $g(x) = 5x^4 + 3x^2 + 9x + 8$.

The main idea of this problem is to notice that $x = 10$ is the only real solution to $f(x) = 1933$.

This can be done by observing that the coefficients of $f(x)$ are simply the digits of 1933.

Alternatively, factoring $f(x) - 1933$ yields $(x - 10)(x^2 + 19x + 193)$, from which we can deduce that the only real solution is $x = 10$.

From here, we can see that $g(10) = 50000 + 300 + 90 + 8 = \boxed{50398}$.

Problem 3-3

The value of

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{50^2}\right)$$

can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

Proposed by: Kat Dou

Solution 3-3

The only way of simplifying such a long sequence is by making some of the terms cancel out. We can combine fractions to get that

$$\left(\frac{2^2-1}{2^2}\right)\left(\frac{3^2-1}{3^2}\right)\left(\frac{4^2-1}{4^2}\right)\cdots\left(\frac{50^2-1}{50^2}\right)$$

Note that the numerators are differences of squares. Thus, the sequence is equal to

$$\left(\frac{(2-1)(2+1)}{2^2}\right)\left(\frac{(3-1)(3+1)}{3^2}\right)\left(\frac{(4-1)(4+1)}{4^2}\right)\cdots\left(\frac{(50-1)(50+1)}{50^2}\right)$$
$$=\left(\frac{(1)(\cancel{3})}{2^{\cancel{2}}}\right)\left(\frac{(\cancel{2})(\cancel{4})}{\cancel{3}^{\cancel{2}}}\right)\left(\frac{(\cancel{3})(\cancel{5})}{\cancel{4}^{\cancel{2}}}\right)\cdots\left(\frac{(\cancel{49})(51)}{50^{\cancel{2}}}\right)$$

Since each part of the numerator can cancel out part of the denominator of the term either before it or after it, we are left with the numbers at the beginning and the end of the sequence.

Our final answer is thus

$$\frac{(1)(51)}{(2)(50)}=\frac{51}{100}\implies\boxed{151}$$

Note that

$$1-\frac{1}{n^2}=\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)=\left(\frac{n-1}{n}\right)\left(\frac{n+1}{n}\right)$$

Therefore,

$$\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{50^2}\right)=\prod_{n=2}^{50}\left(1-\frac{1}{n^2}\right)$$
$$=\prod_{n=2}^{50}\left(\frac{n-1}{n}\right)\cdot\prod_{n=2}^{50}\left(\frac{n+1}{n}\right)$$

Let's take a look at $\prod_{n=2}^{50}\left(\frac{n-1}{n}\right)$. The expression expands to $\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{4}}\cdots\frac{\cancel{48}}{\cancel{49}}\cdot\frac{\cancel{49}}{50}=\frac{1}{50}$.

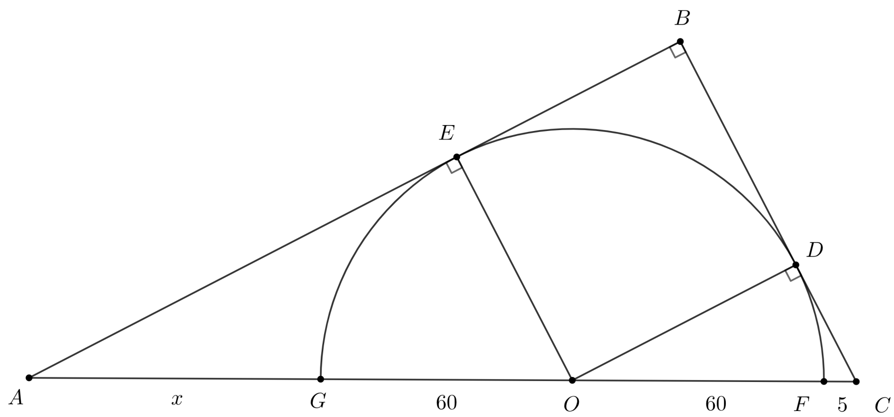
Similarly, $\prod_{n=2}^{50}\left(\frac{n+1}{n}\right)$ expands to $\frac{\cancel{3}}{2}\cdot\frac{\cancel{4}}{\cancel{3}}\cdot\frac{\cancel{5}}{\cancel{4}}\cdots\frac{\cancel{50}}{\cancel{49}}\cdot\frac{51}{\cancel{50}}=\frac{51}{2}$.

Our final answer is therefore

$$\frac{1}{50}\cdot\frac{51}{2}=\frac{51}{100}\implies\boxed{151}$$

Problem 3-4

In the diagram below (which is not to scale), GF is the diameter of a semicircle centered at O. The semicircle is tangent to segment AB at E and BC at D. Triangle ABC is right-angled at B and A, G, O, F and C all lie on the same line. We have $GO = OF = 60$ and $FC = 5$. Find the length of AG, shown as x in the diagram.



Proposed by: Nicole Fan

Solution 3-4

Since EO and OD are radii of the semicircle,

$$EO = OD = OG = 60$$

Looking at $BDOE$, $\angle BEO = \angle DBE = 90°$. Because of the equal lengths we found earlier, $BDOE$ is a square, and

$$BD = BE = 60$$

We now note that $OC = 65$ while $OD = 60$ so ΔDCO is a 5,12,13 right triangle and $DC = 25$. Thus,

$$BC = BD + CD = 60 + 25 = 85$$

Also,

$$AC = x + 60 + 60 + 5 = x + 125$$

Since they share $\angle BAC$ and they are both right triangles, $\Delta AEO \sim \Delta ABC$. This means that

$$\frac{AO}{OE} = \frac{AC}{BC}$$

Plugging in our equations for the lengths above we get that

$$\frac{60 + x}{60} = \frac{125 + x}{85}$$
$$5100 + 85x = 7500 + 60x$$
$$25x = 2400$$
$$x = \boxed{96}$$

Problem 3-5

13 wizards stand in a circle. 5 of these wizards are evil wizards, while the other 8 are non-evil. The evil wizards all simultaneously fire laser beams at some other randomly-chosen wizard. Each of the evil wizards may choose any other wizard, evil or non-evil, and two or more evil wizards may fire their lasers at the same other wizard. Two wizards may shoot each other, but wizards do not shoot themselves.

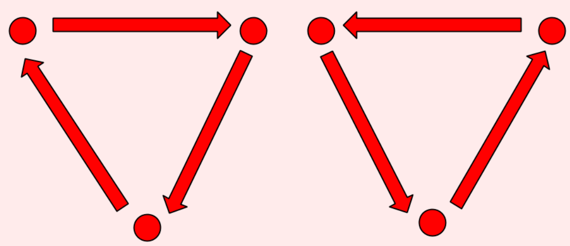
The probability that 3 of the wizards form the vertices of a laser-beam triangle can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Kat Dou

Solution 3-5

We first take the directed graph theoretic interpretation (a laser beam is basically an arrow going from an evil wizard to another wizard).

Now, consider any 3 wizards that form a triangle. There needs to be three arrows in order for each wizard to get "hit", so all three wizards must be evil. In addition, each of these arrows must hit a different person, as there are only 3 of them. Thus, each triangle can only be oriented in two ways as shown in the diagram below.



We now evaluate the answer. Because each evil wizard has out degree at most 1 (shoots at most 1 laser), each evil wizard can be a part of at most 1 triangle. Since there are only $5 < 2 \cdot 3$ evil wizards, there can be at most one triangle.

The total amount of ways the evil wizards can hit other wizards is 12^5, because each evil wizard can hit anyone other than themselves. To count the number of ways to have a triangle of wizards, note that there are $\binom{5}{3} = 10$ ways to choose which 3 wizards form the triangle and 2 ways to orient it. In addition, the other two wizards can hit anyone else without restrictions because they will never form a triangle. Thus, there are 12^2 choices for what the other 2 wizards do. Since all of these cases occur with equal probability, we have that the probability that three wizards hit each other in a triangle is

$$\frac{12^2 \cdot 20}{12^5} = \frac{5}{432}$$

Thus, the answer is $5 + 432 = \boxed{437}$.

Problem 3-6

Real numbers a and b are chosen uniformly at random from the interval $[0, 1]$.

The probability that

$$\left\lfloor \log_2 \left(\frac{1}{a+b} \right) \right\rfloor$$

is odd can be written as $\frac{m}{n}$ where m and n are relatively prime integers. Find $m + n$.

Proposed by: Quinn Chandrashekar

Solution 3-6

We start by seeing that for $\dfrac{1}{a+b}$ to satisfy the condition that $\left\lfloor \log_2 \left(\frac{1}{a+b} \right) \right\rfloor$ is odd, positive integer j must satisfy the inequality

$$\frac{1}{2^{2j}} < a + b < \frac{1}{2^{2j-1}} \tag{1}$$

By taking the reciprocal of that inequality and changing the direction of the signs as a result, we get that

$$2^{2j-1} < \frac{1}{a+b} < 2^{2j}$$
$$\log_2 2^{2j-1} < \log_2 \frac{1}{a+b} < \log_2 2^{2j}$$
$$2j - 1 < \log_2 \frac{1}{a+b} < 2j$$

and since $2j - 1$ is odd, $\left\lfloor \log_2 \left(\frac{1}{a+b} \right) \right\rfloor$ is odd.

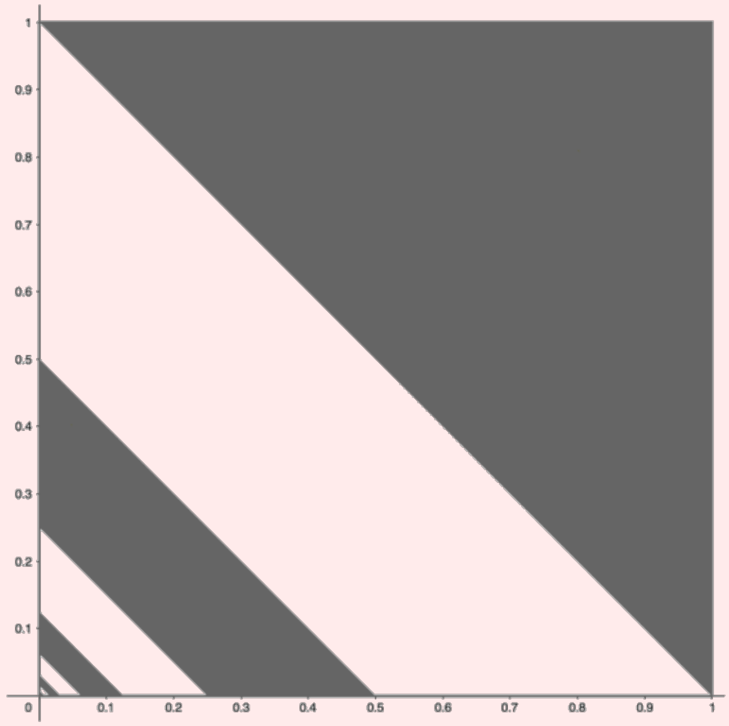
Now let x = a and y = b. Then the sample space can be represented by a 1 by 1 square with vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$. The probability we are searching for is the total area of the regions within the sample space that satisfy (1).

Notice that the triangle bounded by the x axis, the y axis, and the line segment between $(0, 2^j)$ and $(2^j, 0)$ contains only points (x, y) where $x + y < 2^j$, or equivalently,

$$\frac{1}{x+y} > \frac{1}{2^j}$$

This results in infinitely many isosceles right triangles with legs on the x and y axes, along with an additional region beyond the line defined by $j = 0$.

According to (1), points in our sample space which are bounded by a segment defined by $2j - 1$ and a segment defined by $2j$ meet our condition. These zones are shown shaded in the figure below (along with the region beyond $k = 1$ which we will later show to also satisfy our condition).



Let A_j be the area of a triangle with side length $\frac{1}{2^j}$.

Ignoring the area beyond the line defined by $j = 0$, we want to find

$$S = A_1 - A_2 + A_3 - A_4 + \cdots$$

We know that for each progressive triangle, the side length is $\frac{1}{2}$ the previous length, and the area $\frac{1}{4}$ that of the previous triangle. Thus,

$$A_{2j-1} = \frac{1}{4}A_{2j} = \frac{1}{16}A_{2j+1}$$

Since $A_1 = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}$, the sum of all the areas is

$$\begin{aligned}&\left(\frac{1}{8} - \frac{1}{4}\cdot\frac{1}{8}\right) + \frac{1}{16}\left(\frac{1}{8} - \frac{1}{4}\cdot\frac{1}{8}\right) + \frac{1}{16^2}\left(\frac{1}{8} - \frac{1}{4}\cdot\frac{1}{8}\right) + \ldots \\ =&\frac{3}{4}\cdot\frac{1}{8}\left(\sum_{i=0}^{\infty}\left(\frac{1}{16}\right)^i\right)\end{aligned}$$

Using the formula for the sum of an geometric infinite series

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

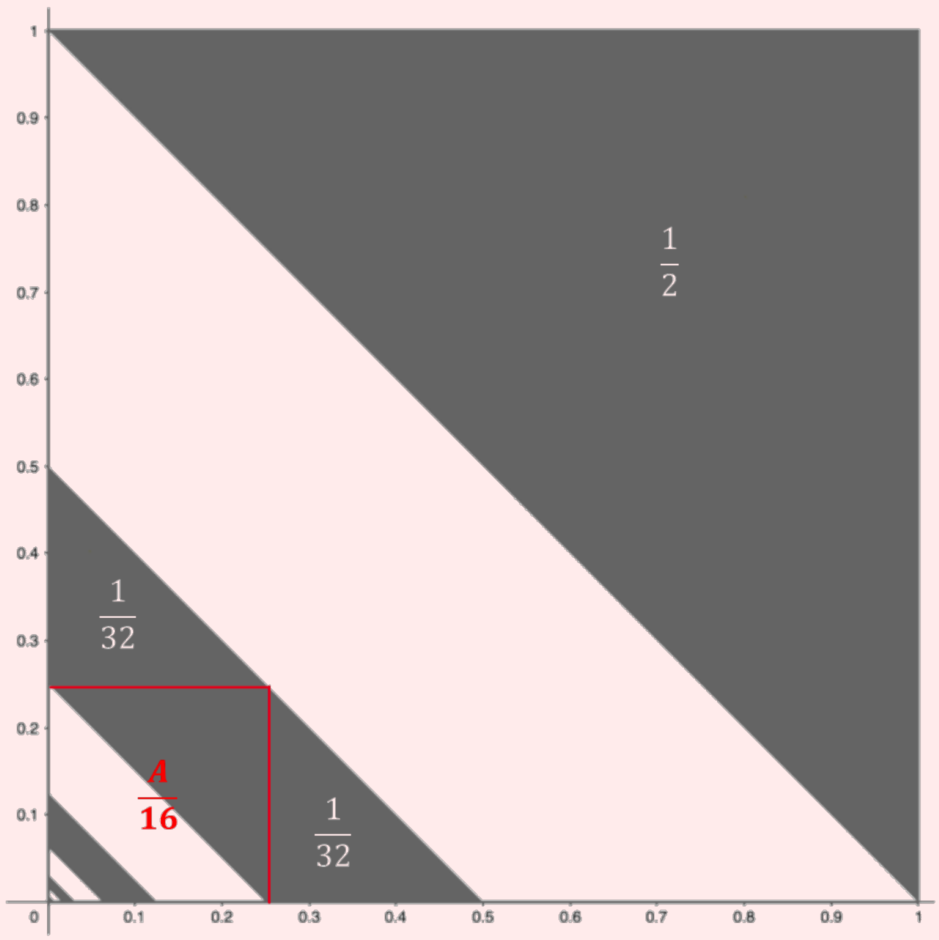
we get the sum of our series to be

$$\frac{3}{4}\cdot\frac{1}{8}\left(1 - \frac{1}{16}\right) = \frac{1}{10}$$

Finally, we deal with the region beyond $k = 1$. We can see clearly from the diagram that the area of this region within our sample space is $1^2 \cdot \frac{1}{2} = \frac{1}{2}$. We note that for any point in this region, $1 < x + y < 2$, which satisfies (1). We thus get the final probability that $\left\lfloor \log_2 \left(\frac{1}{a+b} \right) \right\rfloor$ is odd to be $\frac{1}{10} + \frac{1}{2} = \frac{3}{5}$.

Our final answer is thus $3 + 5 = \boxed{8}$.

Alternatively, consider a portion of the graph in the shape of a square with side length $\frac{1}{4}$ and its bottom left corner at $(0, 0)$. Note that this square is actually identical to the larger square, just scaled down by a factor of 4. Then, if the area of the entire figure is A, the area of this section is $A/16$.



Therefore, we have that

$$\begin{aligned} A &= \frac{A}{16} + \frac{1}{32} + \frac{1}{32} + \frac{1}{2} \\ A &= \frac{1/2 + 1/32 + 1/32}{1 - 1/16} \\ &= \frac{3}{5} \implies \boxed{8} \end{aligned}$$