Anti-Viral 4 Solutions

1 Problems

Problem 4-1

Let a and b be positive integers that have a product of 2019. Compute:

$$(-1)^a \times (-1)^b$$

Solution 4-1

Since 2019 is odd, no matter how you factor it, both of its factors will also be odd. Since $(-1)^{\text{odd}} = -1$, the given expression just reduces to

$$(-1) \times (-1) = \boxed{1}.$$

Problem 4-2

A positive integer is *prime-o* if it has an odd, prime number of factors. How many positive integers less than 100 are *prime-o*?

Solution 4-2

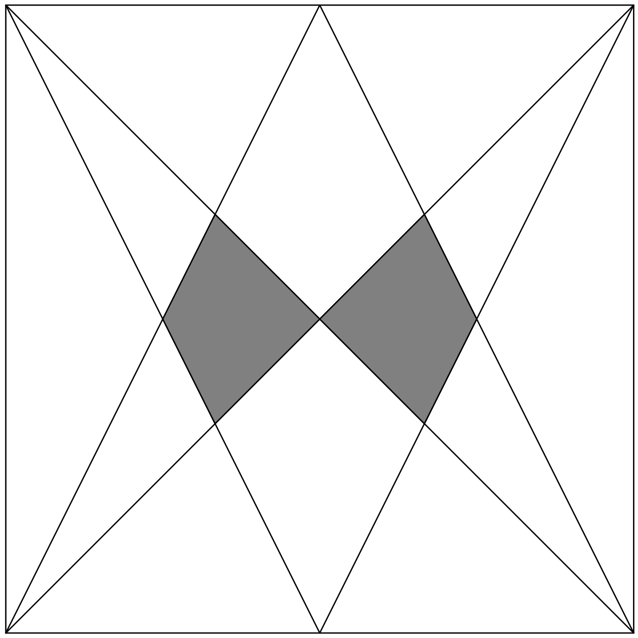
We know that a number n of the form $n = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$ has exactly $(e_1+1)(e_2+1)\cdots(e_n+1)$ factors. So, in order for n to have an odd, prime number of factors, we must have $e_1 = p-1$, $e_2 = \cdots = e_n = 0$, or equivalently, our n must be a prime to the power of one less than a prime (i.e. $p_1^{p_2-1}$).

Since we are bounded by positive integers less than 100, we can just count them:

$$2^2 = 4,\ 2^4 = 16,\ 2^6 = 64,\ 3^2 = 9,\ 3^4 = 81,\ 5^2 = 25,\ 7^2 = 49 \implies \boxed{7}.$$

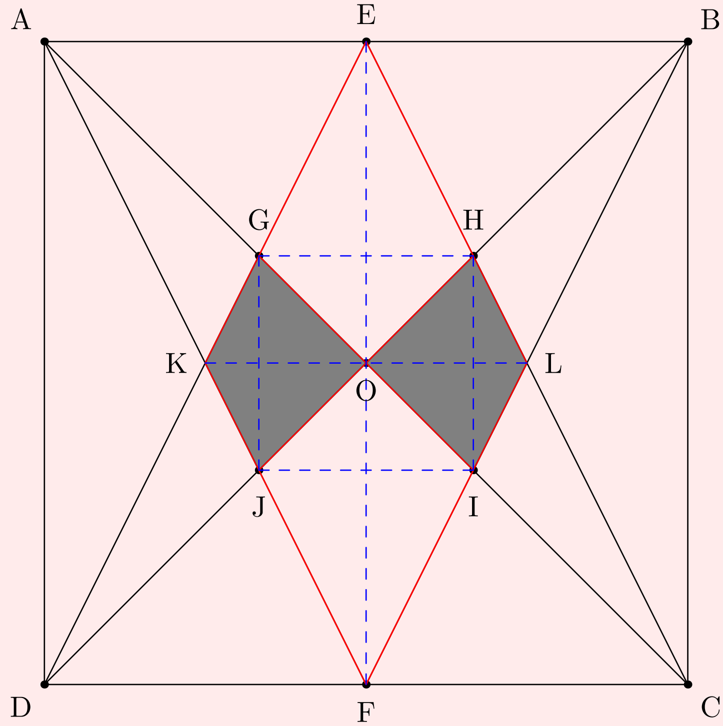
Problem 4-3

In the following square, the shaded region is formed by the intersection of lines joining vertices and midpoints. The shaded region consists of $S\%$ of the entire square. What is the value of $\lfloor S \rfloor$, the greatest integer less than or equal to S?



Proposed by: Ethan Jeon

Solution 4-3



For a quadrilateral whose diagonals d_1 and d_2 are perpendicular, the area is $\dfrac{d_1 d_2}{2}$.

The side of the square can be any length, so we can let the side of the square be 1. To find the area of the shaded region, we just need to find $\overline{GJ}$ and $\overline{OK}$.

Quadrilateral $ELFK$ has diagonals of length $\frac{1}{2}$ and 1, so its area is $\frac{1}{4}$ using the formula. To find the diagonals of the shaded quadrilaterals, we can start by observing that $\triangle GCD$ and $\triangle AEG$ are similar and one of their bases is half the other, so we can say the ratio of their heights is $2:1$, thus $\overline{GJ}$ is equal to $\frac{1}{3}$.

Furthermore, the other diagonals $\overline{OK}$ and $\overline{OL}$ are both $\frac{1}{4}$. Now, we know that the combined area of the shaded regions is

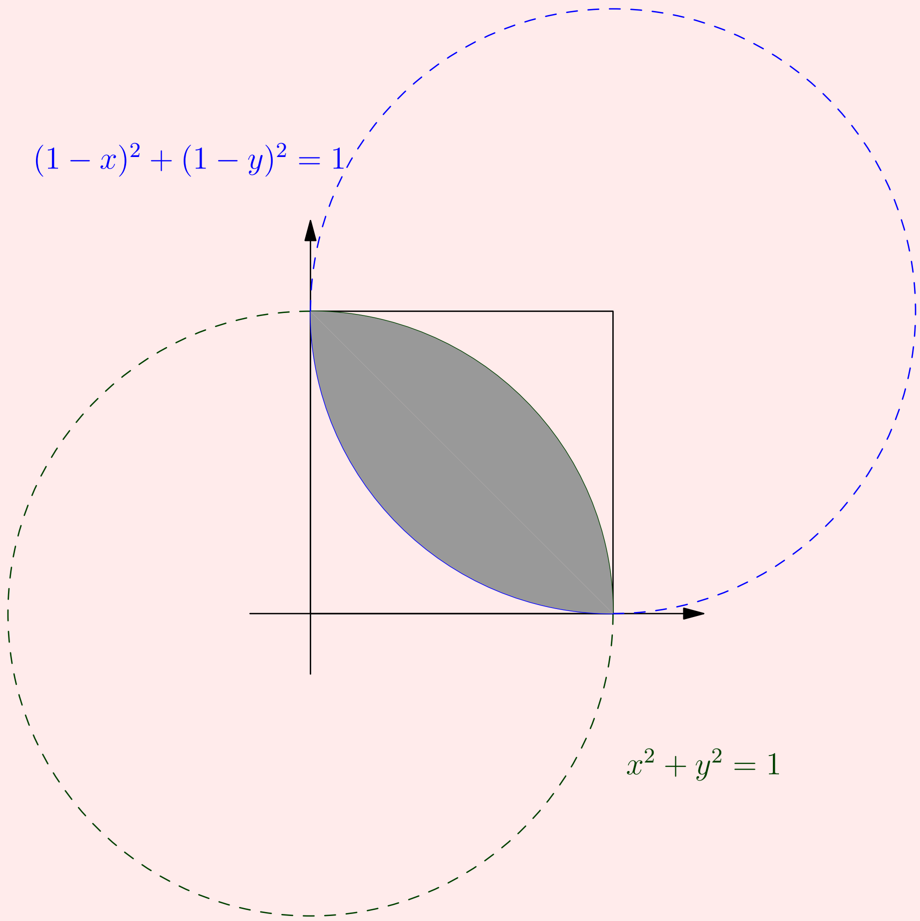
$$2 \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{12} = 8.\overline{3}\% \implies \lfloor S \rfloor = \boxed{8}.$$

Problem 4-4

The variables x and y are chosen uniformly at random from the interval $[0, 1]$. The probability that both $x^2 + y^2 < 1$ and $(1-x)^2 + (1-y)^2 < 1$ are satisfied is p. Find $\lfloor 1000p \rfloor$, which is the greatest integer less than or equal to $1000p$.

Solution 4-4

Graphing the two regions and the region we can select from, we can get the following image:



Note that the area of the square minus the area of a quarter-circle is equal to one of the unshaded areas:

$$A_{\text{unshaded}/2} = A_{\square} - A_{\circ/4} = 1 - \frac{\pi}{4}$$

The shaded area is just the area of the square minus the area of the two unshaded areas:

$$A_{\text{shaded}} = 1 - 2A_{\text{unshaded}/2} = 1 - 2\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{2} - 1 = 0.5707$$

Therefore, $p = 0.5707$ and $\lfloor 1000p \rfloor = \boxed{570}$.

Problem 4-5

Spirited Sam wrote down two arithmetic sequences on the board. Pensive Pam computed the product of corresponding terms of the arithmetic sequence. Careless Cam erased everything except for the first three terms that Pam wrote, which were 310, 455, 624. What was the sixth term in Pam's sequence?

Solution 4-5

Let the two sequences that Sam wrote be a, $a+b$, $a+2b$ and c, $c+d$, $c+2d\ldots$ Then, we have the following system of equations:

$$\begin{aligned} ac &= 310 \\ (a+b)(c+d) &= 455 \\ (a+2b)(c+2d) &= 624 \end{aligned}$$

Expand to get:

$$ac = 310 \tag{1}$$
$$ac + ad + bc + cd = 455 \tag{2}$$
$$ac + 2ad + 2bc + 4cd = 624 \tag{3}$$

Subtract (1) from (2) to get $ad + bc + cd = 145$ — (4).

Subtract (1) from (3) to get $2ad + 2bc + 4cd = 314$.

Divide by 2 to get $ad + bc + 2bd = 157$ — (5).

Subtract (4) from (5) to get $bd = 12$. We also know that $ad + bc = 133$.

Finally, we plug in to find:

$$(a+5b)(c+5d) = ac + 5(ad+bc) + 25bd = 310 + 5(133) + 25(12) = \boxed{1275}.$$

Problem 4-6

The sequence

$$\sum_{i=1}^{2019} \tan(i) \times \tan(i+1) = \tan(1) \times \tan(2) + \tan(2) \times \tan(3) + \cdots + \tan(2019) \times \tan(2020)$$

can be expressed in the form $\frac{\tan(x)}{\tan(y)} - z$, where x, y, and z are positive integers. If all given angles are in radians, determine $x + y + z$.

Proposed by: Edward Xiao

Solution 4-6

The first step to solving this problem is to realize that it's simply impossible without telescoping, so our main focus will be to turn the expression into something that telescopes. If you know your trigonometric identities very well, you might notice that $\tan(a - b) = \frac{\tan(a)-\tan(b)}{\tan(a)\tan(b)+1}$ has a suspiciously helpful numerator. Let's try to make use of it.

$$\tan(1) = \tan((n+1) - n) = \frac{\tan(n+1) - \tan(n)}{\tan(n)\tan(n+1) + 1}$$

So

$$\frac{\tan(n+1) - \tan(n)}{\tan(1)} = \tan(n)\tan(n+1) + 1$$

And

$$\tan(n)\tan(n+1) = \frac{\tan(n+1) - \tan(n)}{\tan(1)} - 1$$

Just like magic, we have transformed the original sequence into a telescoping sequence. Now when we add up all the terms, the numerator will cancel everything except the first and last terms, and we get the expression

$$\begin{aligned}
&= \frac{\tan(2020) - \tan(1)}{\tan(1)} - 2019 \\
&= \frac{\tan(2020)}{\tan(1)} - \frac{\tan(1)}{\tan(1)} - 2019 \\
&= \frac{\tan(2020)}{\tan(1)} - 1 - 2019 \\
&= \frac{\tan(2020)}{\tan(1)} - 2020
\end{aligned}$$

Thus, our desired value is $2020 + 1 + 2020 = \boxed{4041}$.